Anti Reciprocal Inequality.

Problem.

For any $a,b \in \mathbb{R}$ such that $0 < a < b$ find maximal value of

$H(x_1,x_2,\ldots,x_n) := (x_1+x_2+\ldots+x_n)\left(\frac{1}{x_1}+\frac{1}{x_2}+\ldots+\frac{1}{x_n}\right)$

if $x_1,x_2,\ldots,x_n \in [a,b]$.

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For n-tuple $(x_1,x_2,\ldots,x_n)$, where $x_k \in [a,b], k = 1,2,\ldots,n,$ we say that it is a boundary n-tuple if $x_k = a$ or $x_k = b$ for each $k = 1,2,\ldots,n$ and about non-boundary n-tuple $(x_1,x_2,\ldots,x_n)$ we say that it can be improved if there is boundary n-tiple $(c_1,c_2,\ldots,c_n)$ such that

$H(x_1,x_2,\ldots,x_n) \le H(c_1,c_2,\ldots,c_n)$.

Lemma.

Each non-boundary n-tuple $(x_1,x_2,\ldots,x_n) \in [a,b]^n$ can be improved.

Proof.

Let n-tuple $(x_1,x_2,\ldots,x_n) \in [a,b]^n$ is not boundary, i.e. at least one of $x_1,x_2,\ldots,x_n$ isn't equal a and b let it be x_1, i.e. $a < x_1 < b$.

Denoting $\alpha := x_2+\ldots+x_n, \beta := \frac{1}{x_2}+\ldots+\frac{1}{x_n}$ for fixed $x_2,\ldots,x_n \in [a,b]$, we obtain

$H(x_1,x_2,\ldots,x_n) = (\alpha+x_1)\left(\beta+\frac{1}{x_1}\right) = \alpha\beta+1+\beta x_1+\frac{\alpha}{x_1} =$

$\alpha\beta+1+\beta\left(x_1+\frac{p}{x_1}\right)$, where $p = \frac{\alpha}{\beta}$.

We will prove that $x+\frac{p}{x}$ attain its greatest value on the segment $[a,b]$ in the one of the ends of this segment – in a or in b, that is

$$\max_{x\in[a,b]}\left(x+\frac{p}{x}\right) = \max\left\{a+\frac{p}{a}, b+\frac{p}{b}\right\}.$$

Really, suppose that there is $x \in (a,b)$ such that $x+\frac{p}{x} > \max\left\{a+\frac{p}{a}, b+\frac{p}{b}\right\}$,

then $\begin{cases} x+\frac{p}{x} > a+\frac{p}{a} \\ x+\frac{p}{x} > b+\frac{p}{b}\end{cases} \Leftrightarrow \begin{cases}(x-a)(ax-p) > 0 \\ (x-b)(bx-p) > 0\end{cases} \Rightarrow \begin{cases} ax-p > 0 \\ bx-p < 0\end{cases}$

because $a < x < b$. Hence $\frac{p}{a} < x < \frac{p}{b} \Rightarrow \frac{p}{a} < \frac{p}{b} \Leftrightarrow b < a$, that is contradiction with $a < b$.

Thus, $x_1+\frac{p}{x_1} \le \max\left\{a+\frac{p}{a}, b+\frac{p}{b}\right\}$ and $H(x_1,x_2,\ldots,x_n) \le H(c,x_2,\ldots,x_n)$,

where $c := a$ if $a+\frac{p}{a} \ge b+\frac{p}{b}$, else $c := b$.

By the same way we can "repair" each $x_i \notin \{a,b\}$.

Corollary.

$\max\left\{H(x_1,x_2,\ldots,x_n) \mid x_i \in [a,b], i = 1,2,\ldots,n\right\} =$

$\max\{H(x_1,x_2,\ldots,x_n) \mid x_i \in \{a,b\}, i = 1,2,\ldots,n\} =$

$\max\left\{(ka+mb)\left(\frac{k}{a}+\frac{m}{b}\right) \mid 0 \le k,m \in \mathbb{Z}, k+m = n\right\} = n^2+\left[\frac{n^2}{4}\right]\left(\sqrt{\frac{a}{b}}-\sqrt{\frac{b}{a}}\right)^2.$

Proof.

First equality immediately follows from Lemma.

For any boundary n-tuple $(x_1,x_2,\ldots,x_n)$ let $A := \left\{i \mid i \in \{1,2,\ldots,n\} \;\&\; x_i = a\right\}$ and $B := \left\{i \mid i \in \{1,2,\ldots,n\} \;\&\; x_i = b\right\}$ and let $k = |A|, m = |B|$.

Then $k+m = n,\ x_1+x_2+\ldots+x_n = ka+mb$, $\frac{1}{x_1}+\frac{1}{x_2}+\ldots+\frac{1}{x_n} = \frac{k}{a}+\frac{m}{b}$ and, therefore, $\max\{H(x_1,x_2,\ldots,x_n) \mid x_i \in \{a,b\}, i = 1,2,\ldots,n\} =$

$$\max\left\{(ka+mb)\left(\frac{k}{a}+\frac{m}{b}\right) \mid 0 \le k,m \in \mathbb{Z},\ k+m = n\right\}.$$

Since $(ka+mb)\left(\frac{k}{a}+\frac{m}{b}\right) = k^2+m^2+km\left(\frac{a}{b}+\frac{b}{a}\right) = n^2+km\left(\frac{a}{b}+\frac{b}{a}-2\right) =$ $n^2+km\left(\sqrt{\frac{a}{b}}-\sqrt{\frac{b}{a}}\right)^2 \le n^2+\left[\frac{n^2}{4}\right]\left(\sqrt{\frac{a}{b}}-\sqrt{\frac{b}{a}}\right)^2$ (because $\left[\frac{n^2}{4}\right]-km =$ $\left[\frac{n^2-4km}{4}\right] = \left[\frac{(k-m)^2}{4}\right] \ge 0$) and equality occurs if $k = \left[\frac{n+1}{2}\right], m = \left[\frac{n}{2}\right]$ then $\max\left\{(ka+mb)\left(\frac{k}{a}+\frac{m}{b}\right) \mid 0 \le k,m \in \mathbb{Z},\ k+m = n\right\} = n^2+\left[\frac{n^2}{4}\right]\left(\sqrt{\frac{a}{b}}-\sqrt{\frac{b}{a}}\right)^2$

and, therefore,

$$\max\left\{H(x_1,x_2,\ldots,x_n) \mid x_i \in [a,b], i = 1,2,\ldots,n\right\} = n^2+\left[\frac{n^2}{4}\right]\left(\sqrt{\frac{a}{b}}-\sqrt{\frac{b}{a}}\right)^2.$$